GROUP THEORY 2024 - 25, SOLUTION SHEET 5

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. (1) Any element in a finite group is torsion, hence $Tor(A) = A$.

(2) No element except 0 have finite order, so its torsion group is trivial.

(3) Let $[q] \in \mathbb{Q}/\mathbb{Z}$ be any element represented by $q = \frac{a}{b} \in \mathbb{Q}$. Then

$$b[q] = [bq] = [a] = [0] \in \mathbb{Q}/\mathbb{Z}$$

since $a \in \mathbb{Z}$. Hence every element is torsion and $Tor(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

(4) Let $x \in \mathbb{C}^\times$ and write it in polar form $x = re^{i\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$. Then $x^n = r^n e^{in\theta} = 1$ if and only if $r = 1$ and $n\theta = 0 \mod 2\pi$, i.e. $x = e^{2\pi ik/n}$ for $k \in \mathbb{Z}$. Those are the n-roots of unity μ_n. Hence

$$Tor(\mathbb{Q}^\times) = \mu_\infty = \bigcup_{n \in \mathbb{N}_{>0}} \mu_n.$$

(5) We know that subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z} \cong \mathbb{Z}$ which are free, hence without torsion.

(6) We saw in the course that subgroups of finite free abelian group are free abelian, which shows that their torsion subgroup is trivial.

Exercise 3. Since G is finitely generated, there exists a finite set of generators for G. Let $g_1, g_2, \ldots, g_k$ be a set of generators for G, so that every element of G can be written as an integer linear combination of these generators:

$$g = n_1 g_1 + n_2 g_2 + \cdots + n_k g_k,$$

where $a_1, a_2, \ldots, a_k \in \mathbb{Z}$.

Since $\mathrm{Tor}(G) = G$, every element in G is a torsion element. This implies that for each generator $g_i \in G$, there exists a positive integer m_i minimal with the property that $m_i \cdot g_i = 0$ for $i = 1, \ldots, k$ (m_i is the order of g_i).

Since G is generated by the finite set $\{g_1, g_2, \ldots, g_k\}$ and each g_i has finite order m_i, there are only finitely many possible combinations of the generators $g_1, g_2, \ldots, g_k$ with integer coefficients a_i modulo m_i, implying that G itself is finite.

Exercise 4. **(1) $\implies$ (2)**: For all $i \in I$, define $e_i \in \mathbb{Z}^{\oplus I}$ as:

$$e_i := (a_j)_{j\in I} \in \mathbb{Z}^{\oplus I}, \text{ where } a_j = 1 \text{ if } j = i \text{ and } a_j = 0 \text{ if } j \neq i.$$

It is straightforward to show using the definition of direct sums that the set $\{e_i\}_{i\in I}$ is a basis for $\mathbb{Z}^{\oplus I}$. Now if $A \cong \mathbb{Z}^{\oplus I}$ the then homomorphic image of the e_i for a basis for A.

(2) $\implies$ (1): Fix a basis, $(a_k)_{k\in I}$ of A, then every element $x \in A$ can be uniquely written as:

$$x = \sum_{k\in I} n_k a_k$$

for some $n_k \in \mathbb{Z}$. Consider the following function, which is well-defined due to the aforementioned uniqueness:

$$\varphi : A \to \mathbb{Z}^{\oplus I}, \sum_{k\in I} n_k a_k \mapsto (n_k)_{k\in I}.$$

It is a straightforward check to see that φ is an isomorphism of Abelian groups.

Exercise 5. We note that if f extends to a group homomorphism, it must be defined by the formula

$$(a, b, c) \mapsto af(e_1) + bf(e_2) + cf(e_3)$$

and hence it must be unique. On the other hand, one checks that the above formula indeed defines a group homomorphism and therefore there exists a unique group homomorphism $\varphi : F \to \mathbb{Z}^2$ which extends f.

The image of a group homomorphism is always a subgroup of the codomain. Since we saw in the lectures that subgroups of finite free abelian groups are finite free abelian, this answers positively to the question.

Exercise 6. We will constantly use the fact that any subgroup of $\mathbb{Z}^k$ is free of rank $l \leq k$. In each case we will denote the Abelian group in question by A.

(1) Since $\{(1, 1)\}$ is a generating set of A and is linearly independent, it is a basis for A and hence the rank of A is 1.
(2) The rank of A is 1 again since $B = \{(1, 2)\}$ is a basis for A. The set B is linearly independent and generates A as $(-3, -6) = (-3)(1, 2)$.
(3) One checks that $\{1, \sqrt{2}, \sqrt{3}\}$ forms a basis for A and hence the rank of A is 3.
(4) The rank of A is 3 since the three elements generate A and are linearly independent which can be seen by observing that the determinant of the following matrix is non-zero:

$$\begin{pmatrix} 1 & 2 & 1 \\ 5 & 3 & -9 \\ 1 & 8 & 34 \end{pmatrix}.$$

(5) Note that the set $B = \{(1, 5, 1), (2, 3, 8)\}$ is linearly independent and generates A since $(1, -9, 13) = (-3)(1, 5, 1) + 2(2, 3, 8)$. Hence the rank of A is 2.

Exercise 7. The same proof as Proposition 11 of the lecture notes apply to show that $\mathbb{Q}^{>0}$ is not finitely generated. To show that it is free, we show that the set $B = \{p_i | \ p_i \text{ is prime}\}$ of prime numbers forms a basis. Let $q = \frac{a}{b}$ be written in indecomposable form, with $a, b \in \mathbb{N}_*$. Decompose a and b as a product of powers of prime numbers. Note that the prime numbers appearing in each decomposition are distinct since the fraction $\frac{a}{b}$ has been chosen to be indecomposable. Using those decompositions, we obtain q as a finite product of powers of elements of B (the powers are negative for the primes appearing in the decomposition of b). If there was more than one decomposition of q as a product of powers of primes, it would yield distinct decompositions of either a or b (or of both) as product of powers of primes, by seperating the positive and negative powers. By unicity of the decomposition of natural numbers (seen in linear algebra 2) we obtin a contradiction.

We have shown that B is a basis of the abelian group $\mathbb{Q}^{>0}$, which means that it is free by exercise 4.

Exercise 8. Consider the short exact sequence:

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

The induced sequence of torsion subgroups is:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Which is clearly not exact due to the failure of the surjectivity of the map $0 \to \mathbb{Z}/2\mathbb{Z}$.